Kindergartners’ Understanding of Additive Commutativity within the Context of Word Problems

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Baroody and Gannon (1984) proposed that children’s understanding of additive commutativity progresses through several levels of understanding based on a unary view of addition (change meaning) before developing a “true” level of understanding based on a binary conception (part-whole meaning). Resnick (1992) implied that children have both a unary and a binary conception of additive commutativity from the earliest stages of development. Fifty-three 5- and 6-year-old ($M = 6.0$) kindergartners’ unary and binary understanding of additive commutativity was investigated using performance on tasks involving change-add-to and part-part-whole word problems, respectively. The data were inconsistent with the predictions of both models and suggest three alternate theoretical explanations. Moreover, the data indicate that success on a task involving change-add-to problems may be a more rigorous test of understanding of additive commutativity than that involving part-part-whole problems. © 2001 Academic Press

Key Words: addition; commutativity; early childhood; kindergarten; mathematics education; word problems.

Since Ginsburg’s (1977) pioneering work, a number of research reports and theoretical accounts have been published on young children’s understanding of additive commutativity—the concept that the order in which numbers are added does not affect the sum (Baroody, 1982, 1987; Baroody & Gannon, 1984; Baroody, Ginsburg, & Waxman, 1983; Bermejo & Rodriguez, 1993; Cowan & Renton, 1996; Langford, 1981; Resnick, 1983, 1992; Resnick & Ford, 1981; Sophian, Harley, & Martin, 1995; Starkey & Gelman, 1982; for a review of the literature, see Baroody, Wilkins, & Tiilikainen, in press). During this time, an extensive literature on addition word problems has accumulated. Included in this
literature is research on the different types of addition problems (e.g., Carpenter & Moser, 1982; Fuson, 1979, 1992) and the relative difficulty of these problem types (e.g., Carpenter & Moser, 1982, 1984; Ibarra & Lindvall, 1982; Riley & Greeno, 1988; Riley, Greeno, & Heller, 1983). However, no research to date has considered the effects of problem type on children’s ability to understand and apply the commutative property of addition. In this study, we examined whether young children are equally likely to recognize this regularity in the context of change-add-to missing-outcome and part-part-whole missing-whole problems.

In change-add-to problems, physical action is implied: An initial amount is physically increased by adding more of an amount (see, e.g., Problem 1). In contrast, with a part-part-whole problem, no physical action is implied: Both parts of a whole are present initially and their total indicates the whole (see, e.g., Problem 2).

Problem 1: Big Bird had four marbles. Cookie Monster gave him two more marbles. Now how many marbles does Big Bird have?

Problem 2: Big Bird has four marbles in his left hand and two marbles in his right hand. How many marbles does he have altogether?

Change-add-to problems embody what Weaver (1982) called a unary operation, an operation on one number that results in a second. Part-part-whole problems embody what he called a binary operation, an operation on two amounts that results in a third. Citing Brush (1978) and Gellman and Gallistel (1978) as evidence, Fuson (1979) and Weaver (1982) hypothesized that a unary view of addition develops prior to a binary view. Expanding on Weaver’s (1982) work, Baroody and Gannon (1984) proposed that children’s view of additive commutativity progresses through three levels based on a unary understanding of addition (change meaning) before developing a level of “true” understanding based on a binary conception (part-whole meaning). Because of their unary conception of addition, Level 0 children view “four and two more” (4 + 2) and “two and four more” (2 + 4) as different situations and do not recognize that these numbers have the same sum (see Table 1). Because no conceptual constraints prohibit it, Level 1 children may disregard addend order to make computation of sums easier (e.g., they may treat two and four more [2 + 4] as if it were four and two more [4 + 2]). Yet, when explicitly asked if these situations have the same sum, these children may be unsure or even indicate that the situations do not have the same sum. Baroody and Gannon (1984) called this level protocommutativity. Level 2 children recognize that, although four and two more (4 + 2) and two and four more (2 + 4) represent different situations, they have the same outcome. Weaver (1982) called this level of understanding pseudocommutativity. Level 3 children construct a binary view of addition and, along with it, a true mathematical understanding of commutativity (e.g., the parts two and four are interchangeable in forming the whole six).

Resnick (1992) offered a different view of how additive commutativity develops. According to her model, children progress through four kinds of mathematical thinking: protoquantities, quantities, numbers, and operators. Children at the
protoquantities stage reason about amounts without numerical references and recognize part-whole relations such as Part 1 + Part 2 = Part 2 + Part 1. Children at the quantities stage reason about specific numbers in context and recognize, for instance, that three apples and five apples equals five apples and three apples. Children at the numbers stage can reason about numbers in the abstract and recognize that 3 + 5 = 5 + 3. Children at the operators stage can reason about general number principles and recognize that the order of any two addends does not affect the sum (a + b = b + a).

Unlike Baroody and Gannon (1984), however, Resnick (1992) assumed that young children’s thinking involves both a unary and a binary conception of addition. Referring to the quantities level, for example, Resnick (1992) noted: “It is possible to distinguish several different kinds of addition. . . . Addition can mean combining 4 apples and 3 apples or increasing by 5 a set of 20 marbles” (p. 405).

Some existing evidence does not appear to be consistent with either Baroody and Gannon’s (1984) or Resnick’s (1992) position. It follows from the former that young children’s unary conception of addition should enable them to assimilate change-add-to problems more easily than part-part-whole problems, making it likely they would recognize the commutative property earlier in the first context. According to Resnick’s (1992) model, young children should understand change-add-to and part-part-whole problems equally well and, thus, their understanding of commutativity should not be affected by problem type. Several studies, however, indicate that kindergartners’ typically find part-part-whole problems easier to solve than change-add-to problems (e.g., Ibarra & Lindvall, 1982; Riley & Greeno, 1988).

Furthermore, if children have both a unary and binary concept of addition, as Resnick’s (1992) model implies, then commutativity permission underlying the invention of addition strategies that disregard addend order should apply equally to change-add-to and part-part-whole word problems. DeCorte and Vershaffel (1987), however, found that semantic structure of word problems did have a

| Level 0 | Unary conception of addition + no inkling of commutativity (e.g., 4 + 2 viewed as four and two more exclusively and 2 + 4 viewed as two and four more exclusively) |
| Level 1 | Unary conception of addition + protocommutativity (e.g., 2 + 4 viewed as two and four more but treated as if it were four and two more to save computational effort; no explicit recognition, though, that 2 + 4 has the same sum as 4 + 2) |
| Level 2 | Unary conception of addition + pseudocommutativity (e.g., 2 + 4 viewed as two and four more but as having the same sum as 4 + 2, which represents the different situation four and two more) |
| Level 3 | Binary conception of addition + true (mathematical) commutativity (e.g., 2 + 4 and 4 + 2 viewed interchangeably as the cardinality two combined with the cardinality four to form the cardinality six: 2 + 4 = 4 + 2 = 6) |
significant effect on children’s choice of solution strategy. Their participants used a more efficient “counting-on from larger addend” strategy (e.g., for three candies and five more candies, counting: “Five, six [is one more], seven [is two more], eight [is three more]—the answer is eight”) more often for part-part-whole word problems. This finding was attributed to the lack of an addend order constraint associated with such problems. Although DeCorte and Verschaffel (1987) did not explicitly investigate additive commutativity, their finding also suggests that the Baroody and Gannon (1984) model is inaccurate. It may be the case that children construct a binary view of commutativity before protocommutativity (Level 1 in Table 1) and pseudocommutativity (Level 2 in Table 1).

In this study, we tested Baroody and Gannon’s (1984) and Resnick’s (1992) models in light of the DeCorte and Verschaffel (1987) results. Children were given two commutativity tasks, one in the context of change-add-to word problems and the other in the context of part-part-whole word problems. According to the Baroody and Gannon (1984) model, some young children might not perform successfully on either commutativity task, some should be successful on the change-add-to version only, and some should be successful on both versions. According to the Resnick (1992) model, children should be either unsuccessful or successful on both versions.

METHOD

Participants

Fifty-three children (26 girls and 27 boys) ranging in age from 5 years 6 months (5-6) to 6 years 8 months (6-8) (M = 6-0) participated in the study. The participants were from four kindergarten classes: one in a school serving a predominantly White urban neighborhood (13 students), one serving a predominantly African American urban neighborhood (12 students), and two serving a predominantly White rural neighborhood (28 students). All children had their parents’ or guardians’ permission to participate in the study.

Tasks

Commutativity Tasks

Materials. In both the change-add-to and part-part-whole commutativity tasks, sets of toys were composed of either marbles, crayons, or baseball cards. Marbles were presented in small boxes with a removable lid. Crayons were presented in 10-cm-tall plastic cups. Baseball cards were presented in envelopes. These containers were used so that objects could be shown and then hidden to prevent counting. Because children in a pilot study had trouble remembering the numbers in the word problems, we decided that visual aids were necessary. For the change-add-to version, 7.5-cm × 12.5-cm cards with 2.5-cm numerals were used to cover the added amounts. For both versions, each box, cup, and envelope had a 2.5-cm-tall numeral representing the number of objects in the container. Two stuffed toys approximately 22-cm tall representing the Sesame Street characters Big Bird and Cookie Monster were used as “owners” of the toys.
Procedure for change-add-to version. Change-add-to unknown-outcome word problems (see Table 2) were read to a child and acted out by the experimenter. For example, for the first problem in Table 2, the experimenter said, “Big Bird has nine marbles in a box,” while opening the box (with the numeral 9 on its top) to show the child the nine marbles. He then closed the lid of the box and set it in front of Big Bird. Then the experimenter said, “and I give him seven more marbles,” while he placed seven marbles to the right side of the box and a 7.5-cm × 12.5-cm card with the numeral 7 was placed over the pile of marbles. Next, the experimenter said, “Cookie Monster has seven marbles in a box,” and the box (with the numeral 7 on the top for the child to see) was opened to show the child the seven marbles and then closed and set in front of Cookie Monster. Then the experimenter said, “and I give him nine more marbles,” and nine marbles were physically placed to the right side of the box and a 7.5-cm × 12.5-cm card with the numeral 9 was placed over the pile of marbles. Then the experimenter asked the child: “Do they have the same number of marbles?” “Does Cookie Monster have more marbles?” “Does Big Bird have more marbles?” Or, “You can’t tell who has more marbles?”

Children often explained their responses. However, participants who did not give explanations were randomly asked to explain why they gave particular responses.

Procedure for part-part-whole version. Part-part-whole unknown-outcome word problems (see Table 2) were read to the child and acted out by the experimenter. For instance, for the fourth problem in Table 2, the experimenter said, “Cookie Monster has seven marbles in one box,” and the box (with the numeral 7 on the top of the box for the child to see) was opened to show the child the seven marbles and then closed and set in front of Cookie Monster. Then the experimenter said, “and 15 marbles in another box,” and a second box (with the numeral 15 on the top) was opened to show the child the 15 marbles and then closed and placed to the right of the first box in front of Cookie Monster. Then the experimenter said, “Big Bird has 15 marbles in one box,” and the box (with the numeral 15 on the top of the box for the child to see) was opened to show the child the 15 marbles and then closed and set in front of Big Bird. Then the experimenter said, “and seven marbles in another box,” and a second box (with the numeral 7 on the top) was opened to show the child the seven marbles and then closed and placed to the right of the first box in front of Big Bird. Then the experimenter asked the child: “Do they have the same number of marbles?” “Does Cookie Monster have more marbles?” “Does Big Bird have more marbles?” Or, “You can’t tell who has more marbles?”

Children who did not give explanations for their choices were randomly asked to explain why they chose their response.

Scoring. For both the change-add-to and part-part-whole versions, a child was scored as correct on a commuted trial if the child answered “the same.” As in the Cowan and Renton (1996) study, noncommuted trials were scored in two ways. Using a liberal criterion (Baroody & Gannon, 1984; Cowan & Renton, 1996;
Sophian et al., 1995), a child was scored as correct on such trials if the child gave any response other than “the same.” This included correctly indicating which muppets had more, incorrectly indicating this, or responding with: “I can’t tell.” The latter two (incorrect) responses were rare and, in some cases, would make sense (e.g., if a child incorrectly considered the 12 in 12 + 12 to be larger than the 17 in 17 + 17 or was unsure whether 17 was greater than 12). Using a conservative criterion (Cowan, in press; Cowan & Renton, 1996), a child was scored as correct on a non-commuted trial if the child correctly decided which of the two sums was larger.

In each version, then, participants were assessed on their recognition of commuted pairs and on their ability to selectively apply the commutative property. For the four commuted pairs a child had a one-in-four chance of answering correctly (i.e., .25). Based on the liberal criterion, for the two noncommuted pairs a child had a three-in-four chance of answering correctly (i.e., .75). Children were considered successful on a commutativity task if they correctly answered at an above chance level (i.e., on five or six of the trials, \( p = .03 \), Binomial Test). ¹

**Addition Tasks**

To ensure that participants had to compute sums rather than recall them, in a precession, they were presented with six word problems containing addends similar in size to those used in the commutativity tasks. None of the participants consistently used recall correctly on these items.

Procedures

During a preliminary familiarization session, the experimenter played mathematics games involving number-concept tasks with each child. In addition to

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¹ This value was figured as follows: The probability of getting all six trials correct is \((.25)^4(.75)^2\). Getting five trials correct can happen in six different ways: 12345, 12346, 12356, 12456, 13456, and 23456. The probability of 12345 is \((.25)^4(.75)^2\); of 12346 is \((.25)^4(.75)^2\); of 12356 is \((.25)^4(.75)^2\); of 12456 is \((.25)^4(.75)^2\); of 13456 is \((.25)^4(.75)^2\); and of 23456 is \((.25)^4(.75)^2\). Summing these probabilities results in a probability of 0.030029; thus, \( p = .03 \).
familiarizing students with the experimenter, assessment with these number-concept tasks provided the children’s teacher with information on oral count, numeral reading, enumeration skills, understanding of number after, and small number comparisons. During a second session, children were given addition tasks similar to those to be used in the commutativity tasks. These tasks tested for the presence of addition strategies and the ability to recall sums similar to the target sums.

Twelve trials were used for the commutativity tasks and divided into two sets of six trials. Each set of six trials consisted of four commuted trials (each containing a pair of commuted problems) and two noncommuted trials (see Tables 2 and 3). Children were not asked to solve the 12 pairs of problems, but instead asked if, in each situation, commuted and noncommuted pairs would result in the same sum. In other words, children were asked if Big Bird and Cookie Monster had the same number of toys or a different number.

The noncommuted trials served to deter and to detect a response bias. Twenty-two children classified as having no or incomplete understanding of commutativity answered incorrectly “the same” on only two noncommuted trials. Similarly, 31 children who were classified as having complete understanding missed only one noncommuted trial (and it was due to a misunderstanding of the problem). These results would suggest that the children for the most part were attentive and on task and that the measures were quite reliable.

During the third session, six trials were administered to each child; the other six were administered in the fourth session. The two sets of trials were rotated by problem type, each trial having both a change-add-to and a part-part-whole form (again see Table 2), and by order of administration. To control for order effects, 27 of the subjects were administered a change-add-to version first, and the other

<table>
<thead>
<tr>
<th>Pairs</th>
<th>Context of Problem</th>
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<tbody>
<tr>
<td>Problem Set 1</td>
<td></td>
</tr>
<tr>
<td>9 + 7</td>
<td>7 + 9</td>
</tr>
<tr>
<td>5 + 8</td>
<td>8 + 5</td>
</tr>
<tr>
<td>7 + 4</td>
<td>4 + 7</td>
</tr>
<tr>
<td>7 + 15</td>
<td>15 + 7</td>
</tr>
<tr>
<td>4 + 9</td>
<td>4 + 4</td>
</tr>
<tr>
<td>12 + 12</td>
<td>17 + 17</td>
</tr>
<tr>
<td>Problem Set 2</td>
<td></td>
</tr>
<tr>
<td>6 + 9</td>
<td>9 + 6</td>
</tr>
<tr>
<td>8 + 4</td>
<td>4 + 8</td>
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<tr>
<td>5 + 7</td>
<td>7 + 5</td>
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<tr>
<td>13 + 8</td>
<td>8 + 13</td>
</tr>
<tr>
<td>10 + 10</td>
<td>16 + 16</td>
</tr>
<tr>
<td>3 + 9</td>
<td>3 + 3</td>
</tr>
</tbody>
</table>
26 subjects were administered a part-part-whole version first. There were no significant order effects found. Within sessions, the six trials were administered in random order.

RESULTS

The success on the change-add-to and part-part-whole commutativity tasks, using the liberal scoring criterion for noncommuted trials, is summarized in Table 4. Contrary to the prediction that follows from the Baroody and Gannon (1984) model, all the data did not fall in cells A, B, and D of this table. Moreover, of the 10 participants with an incomplete understanding (see cells B and C), 9 out of these 10 were successful on the part-part-whole version, which is sufficient to reject the model (Binomial Test, \( p = .021 \)). The correlation between student success on the two versions was moderate (\( \phi = .61, p < .001 \)). However, contrary to the prediction that follows from Resnick’s (1992) model, not all the data fell in cells A and D. In other words, the 10 out of 53 participants (18.9%) in cells B and C is significantly different from zero (.95 CI = [.08, .29]) and sufficient to reject this model.

Using the conservative criterion for scoring noncommuted trials changed the classification for only 2 of the 53 participants and did not substantially change the results. In fact, applying this criterion resulted in two additional entries to the critical Cell C in Table 4 (and two less to the noncritical Cell A). In effect, whether we used a liberal criterion, which may overestimate commutativity competence, or the conservative criterion, which may underestimate this competence, the results were statistically significant. These results are similar to those Cowan and Renton (1996) obtained with different commutativity tasks. These researchers, likewise, found that children were only slightly (and not significantly) more successful when a liberal criterion was used. For this reason, only the results using this criterion will be reported.

TABLE 4
Success on Change-Add-To and Part-Part-Whole Versions of the Commutativity Tasks, Using the Liberal Criterion for Noncommuted Trials

<table>
<thead>
<tr>
<th></th>
<th>Successful</th>
<th>Unsuccessful</th>
</tr>
</thead>
<tbody>
<tr>
<td>Change-Add-To</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Successful</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td></td>
<td>31</td>
<td>1</td>
</tr>
<tr>
<td>Unsuccessful</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>12</td>
</tr>
</tbody>
</table>

Note. Cell A is classified as complete understanding of commutativity and cell D as no understanding, and cells B and C represent incomplete understandings. The 10 of 53 subjects (18.9%) in cells C and B is significantly different from Resnick’s predicted zero (.95 CI = [.08, .29]). Moreover, the 9 subjects in cell C are sufficient to reject the Baroody and Gannon model (Binomial test, \( p = .021 \)).
Table 5 presents the mean number of correct trials for each version of the commutativity task for the children in each of the four cells in Table 4. The nine participants who were successful on only the part-part-whole version (Cell C) were, on average, correct more than twice as often on this task as on the change-add-to version, and none of these children were successful on more than one half of the change-add-to trials. A one-way repeated-measures analysis of variance for the two tasks produced a significant task effect \( F[1, 8] = 134.56, p < .001 \). These findings provide clear evidence that these children recognized commutativity in one situation but not in the other and that the childrens’ lack of success on the change-add-to trials was not merely due to measurement error. The results of children who were unsuccessful on both versions (Cell D) were also clear-cut. With the exception of one child, none answered correctly more than one half the trials for either task, and most of the participants answered only two trials correctly. However, the results of the one Cell B child who was successful on only the change-add-to task were less clear. Although her four correct answers on the part-part-whole version may well have been due to chance alone, she almost met criterion for success on this task. Nevertheless, the results overall indicated that the measures reliably detected an understanding of commutativity.

A one-way repeated measures analysis of variance for the two commutativity tasks produced a significant effect \( F[1, 52] = 15.06, p < .001 \). This confirms that the participants more readily recognize commutativity on the part-part-whole version than on the change-add-to version (overall means = 5.07 and 4.38, respectively).

Some secondary results of this study were interesting and gave further validation to the study itself. The 58% of the children who were classified as having a complete understanding of commutativity at the quantities level is consistent with several other studies. Cowan and Renton (1996) found that 13 of 24 (54%) 5-year-olds were successful on a quantities level commutativity task very similar to the unary task used in this study. Baroody and Gannon (1984) found that 51% of 35 kindergartners were successful overall on two separate commutativity tasks. Similarly, the 75% of the children who were successful on the binary task in this study is consistent with the 24 of 30 (80%) 5- to 6-year-olds successful on a similar task (Langford, 1981).

DISCUSSION AND CONCLUSIONS

Theoretical Implications

Contrary to Baroody and Gannon’s (1984) model, children with an incomplete understanding of commutativity were not more successful on the change-add-to version of the commutativity task than on the part-part-whole version. In fact, a significant number of children with an incomplete understanding were more successful on the latter. Contrary to Resnick’s (1992) model, a significant number of children did have an incomplete understanding of commutativity. That is, some children were successful on only one of the commutativity tasks.
The discordant results of this study can be reconciled with Baroody and Gannon’s (1984) model by postulating that young children assimilate part-part-whole problems to their unary conception of addition but more weakly than they do change-add-to problems. In other words, part-part-whole problems are similar enough to change-add-to problems that the former can be understood as addition problems by children with a unary conception of the operation. However, unlike change-add-to problems, the wording of part-part-whole problems does not invoke the order constraint implied by a unary view. Put differently, the inconsistent responders in this study are analogous to Baroody and Gannon’s (1984) participants at the protocommutativity level (Level 1 in Table 1). The latter likewise treated, for example, $3 + 5$ and $5 + 3$ as equivalent in some circumstances (where it saved computational effort) but, when explicitly asked if $3$ and $5$ more were equal to $5$ and $3$ more (the order constraint of the unary conception was directly invoked), they were unsure of the equivalence.

The discordant results of this study can also be reconciled with Resnick’s (1992) model. It is quite possible that our inconsistently correct children did have both a unary and a binary understanding of addition but that these conceptions were only weakly linked. In effect, these children assimilated the change-add-to problems to the first conception and, because of its implied order constraint,
denied commuted problems were equivalent. They assimilated part-part-whole problems to their unary conception and, because of the lack of an order constraint, considered commuted problems equivalent.

A third explanation is that elements of both models are correct (see Table 6). Children may initially construct a unary conception of addition and—like our participants who were unsuccessful on both versions of the commutativity task—may have no understanding of the principle (Level 0 in Table 6). They may then construct a binary conception of addition that is only loosely connected to their unary conception. This might permit children—like some of the participants in DeCorte and Vershaffel (1987) who more often counted-on from the larger addend in the context of part-part-whole problems—to view order as a constraint in some situations and as irrelevant in others (Level 1 in Table 6). Students at this level may have only an implicit understanding of commutativity as a tool that allows them to economically disregard addend order, yet have no explicit understanding of commutativity. It may also be the case that some children in the DeCorte and Vershaffel (1987) study, as well as the inconsistent participants in this study, may explicitly recognize the commutative relation in part-part-whole situations but not in change-add-to situations (Level 2 in Table 6). These participants may still be bound by the constraints of a unary schema, yet fully understand the concept when they can assimilate a situation to their unconstrained binary schema. At some juncture, children—like our participants who were consistently correct (Level 3 in Table 6)—must recognize that even commuted unary situations are equivalent, and this insight may help to further connect this conception of addition with their binary view.

Even in adults, such statements as \(4 + 2 = 6\) and \(2 + 4 = 6\), which are mathematically equivalent, psychologically may imply different meanings (Kaput, 1979). That is, adults may view \(4 + 2 = 6\) and \(2 + 4 = 6\) in unary (change-add-to) terms, as implying different situations and yet recognize that these different situations produce the same result (cf. Kaput, 1979). Perhaps a less pejorative term for this understanding than “pseudocommutativity” might be unary commutativity (see Level 3 in Table 6). Adults may also view \(4 + 2 = 6\) and \(2 + 4 = 6\) in binary (part-part-whole) terms, as implying the parts 4 and 2 form the whole 6 regardless of how they are combined. This view is closer to the mathematical

<table>
<thead>
<tr>
<th>Level</th>
<th>Proposed Levels of Commutativity Development</th>
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<tbody>
<tr>
<td>Level 0</td>
<td>Unary conception of addition + no inkling of commutativity</td>
</tr>
<tr>
<td>Level 1</td>
<td>Unary and binary conception of addition + protocommutativity</td>
</tr>
<tr>
<td>Level 2</td>
<td>Unary conception of addition + protocommutativity and binary conception + part-whole commutativity</td>
</tr>
<tr>
<td>Level 3</td>
<td>Unary conception + Change-Add-To commutativity (“pseudocommutativity”) and binary conception of addition + true (mathematical) commutativity</td>
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</table>
definition of addition and commutativity but is not psychologically superior to a unary view or unary commutativity.

Further research is needed to decide which of three alternative explanations offered, if any, best describes children’s development. An important extension of this study would be to combine it with the Decorte and Verschaffel (1987) study and explicitly track the parallel development of children’s understanding of commutativity and their choice of addition problem-solving strategies. This may allow teasing apart those students who count-on from the larger addend because of their understanding of commutativity (Level 2) and those that employ this method as a labor-saving device, yet have no true understanding of commutativity (Level 1).

In any case, the three models described are consistent with viewing children’s early arithmetic development in terms of Anderson’s (1984) model of weak versus strong schema. A weak schema is a generalization that is narrow in scope (few or weak connections among concepts) and perhaps lacking logical coherence. In the third account, for instance, Level 0 and Level 1 children’s unary and binary views are only loosely connected and underlie logically inconsistent responses to commutativity tasks with different (change-add-to versus part-part-whole) wording. The next two levels (Level 2 and 3 in Table 6) would represent progress toward a strong addition schema, a relatively broad and logically coherent generalization.

The second and third alternative explanations, particularly, are consistent with the hypothesis that children have multiple, loosely connected part-whole schemas (Baroody et al., in press). Unlike Resnick’s (1992) view of a unitary part-whole schema with a single set of rules, including commutativity, children may have different sets of generalizations about part-whole relations, some of which order information is unimportant and others of which such information is important. One of the tasks children face, then, is sorting out which arithmetic situations order is and is not important.

**Methodological Implications**

Although Baroody and Gannon’s (1984) model suggests that a complete or deep understanding of commutativity is not achieved until children construct a binary concept and can apply the principle to part-part-whole contexts, the results of this study indicate that it is not achieved until they can apply the principle to change-add-to contexts. Success on a commutativity task involving part-part-whole situations represents, at best, a weak permission of commutativity based on the lack of addend-order information. Success on a commutativity task involving change-add-to situations requires applying the principle in spite of explicit addend-order information and thus may be a more rigorous test of it.

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Received May 29, 1998; revised April 7, 2000